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TECH MEMO
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A SEQUENTIAL PROCEDURE FOR SUCCESSIVELY IMPROVING

SAMPLING DISTORTIONS, PRELIMINARY REPORT

This paper is written in the language of mathematical statistics. Expositions for practitioners in Monte Carlo will appear elsewhere.

Let

(1)
$$\zeta_1, \ldots, \zeta_n, \text{ and } \zeta$$

be independent variables (the discussion applies if these ζ's are random vectors, and in applications we are interested in vectors, but for ease of exposition we shall speak of random variables). We consider the real function of $n+1$ real variables

(2)
$$y = y(x_1, \ldots, x_n; x).$$

We are interested in

(3)
$$E(y),$$

the expected value of y as a function on the probability space of the ζ's.

Let

(4)
$$\zeta_i', \quad i = 1, \ldots, n.$$

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be random variables such that for every measurable set A, the probability that ζ_i' is in A is zero if and only if the probability that ζ_i is in A is also zero. Thus with probability 1, a sample value from either of ζ_i and ζ_i' could have arisen as a sample value from the other.

We shall define weight functions

$$w_i'(x_1,\ldots,x_n;x), \quad i=1,\ldots,n$$

such that

$$(5) \qquad E(y) = E'\left(y\prod_{i=1}^{n} w_i'\right)$$

where E' denotes expected value under the probability measure of $\zeta_1',\ldots,x_n',x$. If μ_i and μ_i' are the probability measures for the unprimed and primed random variables, respectively, we take

$$(6) \qquad w_i' = \frac{d\mu_i}{d\mu_i'} .$$

Let us now indicate what we are trying to do. We are given (1) and (2). We wish to estimate (3). This estimate could be made by Monte Carlo analysis, that is, by means of simple random sampling from the joint distribution of the random variables (1). However in the light of (5), we would like to define random variables (4) such that the variance of $y\prod w_i'$ over the probability space of the primed random variables is less than that of y over the space of the unprimed variables. We shall speak of the ζ_i' as the distorted variables and the ζ_i as the undistorted variables. Thus, in estimating $E(y)$ we distort distributions of random variables to achieve greater efficiency.

Let us indicate further the motivation of the analysis to be developed below. Suppose that the probability density of ζ_1 is $\lambda \exp(-\lambda x)$, with x nonnegative. Suppose that y is practically zero unless x_1 is very large. Then in undistorted sampling one would require a large sample because only sample values in the tail of the distribution of ζ_1 give much information. This suggests that one might introduce ζ_1' with probability density $\lambda' \exp(-\lambda'x)$ with λ' smaller than λ; in this way one would get more sample values located in the tail of the distribution of ζ_1. But the choice of λ' is not easy. Let us assume that a value of λ' was selected and a sample drawn from the distorted distributions, and that this selection and this sampling were done prior to the discussion of the present memorandum. Our objective below is to use the sample data to determine a new λ', say λ'' which will yield greater efficiency than λ'.

Let us write (still digressing)

(7) $$\lambda'' = \lambda + (1+\alpha)(\lambda'-\lambda).$$

We shall seek a good value to use for α. We note that (6) with the subscript i deleted specializes to

(8) $$w'' = \frac{\lambda \exp(-\lambda x)}{\lambda'' \exp(-\lambda''x)} \quad ,$$

and that there is a similar expression with primes in place of the double-primes. For purposes to come to light below we calculate using (7) and (8) that

$$w'' = \frac{\lambda}{\lambda+(1+\alpha)(\lambda'-\lambda)} \exp[(\lambda''-\lambda)x]$$

$$= \frac{1}{1+(1+\alpha)\frac{\lambda'-\lambda}{\lambda}} \exp[(1+\alpha)(\lambda'-\lambda)x]$$

$$\dot{=} \frac{1}{\left[1 + \frac{\lambda' - \lambda}{\lambda}\right]^{1+\alpha}} \quad \exp\left[(1+\alpha)(\lambda'-\lambda)x\right],$$

if $\lambda' - \lambda$ is sufficiently small,

$$= \left\{\frac{\lambda}{\lambda'} \exp\left[(\lambda'-\lambda)x\right]\right\}^{1+\alpha}$$

$$= (w')^{1+\alpha}$$

(see (8) with primes in place of the double primes).

The significance of this relation is the following. The difference $\lambda'-\lambda$ measures the amount of distortion in the use of λ' in place of λ. If the amount of the distortion is increased by a small amount $o(\lambda'-\lambda)$, the weight function is raised to the power $1+\alpha$ approximately. This same approximate power law holds for distortions of other commonly used distributions.

Let us now return to the serious development of the paper. We assume that corresponding to each set $(\alpha_1, \ldots, \alpha_n)$ of real numbers near $(0, \ldots, 0)$ there is a set of random variables

$$\zeta_i'', \quad i = 1, \ldots, n,$$

such that for $\alpha_i = 0$, $i = 1, \ldots, n$, the double-primed variables reduce to the ζ_i', and such that the corresponding weight functions satisfy

$$(9) \qquad\qquad w_i'' = (w_i')^{1+\alpha_i}$$

where, as above, the w_i' are the weight functions for the ζ_i'.

We shall estimate (5). Taking into account (5), we write

$$E(y) = \int y \prod_{i=1}^{n} w_i'' \, d\mu_i''$$

where μ_i'' is the probability measure for ζ_i''. From (6) and the similar relation with primes replaced by double-primes, we obtain

$$w_i' \, d\mu_i' = w_i'' \, d\mu_i''$$

which together with (9) gives

$$d\mu_i'' = (w_i')^{-\alpha_i} \, d\mu_i' \, .$$

The above integral for $E(y)$ reduces to

$$E(y) = \int y \prod w_i'' (w_i')^{-\alpha_i} \, d\mu_i' \, ,$$

which with further use of (9) becomes

$$E(y) = \int y \prod w_i' \, d\mu_i' \, .$$

But this is nothing but (5).

What we are interested in is the variance of $t = y \prod w_i''$, which by calculations like those just completed is obtained from (we introduce notation used below)

$$T \equiv E''(t^2) \equiv E'' \left[\left(y \prod w_i'' \right)^2 \right] = \int y^2 \prod \left(w_i'' \right)^2 \left(w_i' \right)^{-\alpha_i} \, d\mu_i'$$

(10)
$$= \int y^2 \prod \left(w_i' \right)^{2+\alpha_i} \, d\mu_i' \, .$$

Since the expected value of t is constant with respect to the α_i, our objective will be to minimize $E''(t^2)$. The utility of the last expression results from the fact that it involves the weights and measures of random variables which

have already been sampled. Thus using the sample from the ζ_i' we shall be able to pick out a good values of the α_i to be used in further sampling.

In practice the relation (9) is only usable when the α_i are small. Hence the same is true of (10). We shall calculate the derivatives of (10) with respect to the α_i at $\alpha_i = 0$, $i=1,\ldots,n$, and use a quadratic approximation for (10).

Differentiation under the integral sign leads to the following derivatives which have been evaluated at $\alpha_i = 0$. Let T denote $E''(t^2)$.

$$c_j \equiv \frac{\partial T}{\partial \alpha_j} = \int y^2 \log w_j' \prod (w_i')^2 \, d\mu_i' ,$$

$$c_{jj} = \frac{\partial^2 T}{\partial \alpha_j^2} = \int y^2 (\log w_j')^2 \prod (w_i')^2 \, d\mu_i' ,$$

$$c_{jk} = \frac{\partial^2 T}{\partial \alpha_j \partial \alpha_k} = \int y^2 \log w_j' \log w_k' \prod (w_i')^2 \, d\mu_i' .$$

We have

$$T = T \Big|_{(\alpha_1,\ldots,\alpha_n) = (0,\ldots,0)} + \sum c_j \alpha_j + \frac{1}{2} \sum \sum c_{jk} \alpha_j \alpha_k .$$

The minimum of T occurs at the solution of

$$c_j + \sum c_{jk} \alpha_k = 0, \qquad j = 1,\ldots,n$$

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